UTTG-20-95
hepth/9511230
29 November 1995

On the Connectedness of the Moduli Space of Calabi–Yau Manifolds

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ABSTRACT

We show that the moduli space of all Calabi–Yau manifolds that can be realized as hypersurfaces described by a transverse polynomial in a four dimensional weighted projective space, is connected. This is achieved by exploiting techniques of toric geometry and the construction of Batyrev that relate Calabi–Yau manifolds to reflexive polyhedra. Taken together with the previously known fact that the moduli space of all CICY's is connected, and is moreover connected to the moduli space of the present class of Calabi–Yau manifolds (since the quintic threefold $\mathbb{P}_4[5]$ is both CICY and a hypersurface in a weighted $\mathbb{P}_4$), this strongly suggests that the moduli space of all simply connected Calabi–Yau manifolds is connected. It is of interest that singular Calabi–Yau manifolds corresponding to the points in which the moduli spaces meet are often, for the present class, more singular than the conifolds that connect the moduli spaces of CICY's.

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1. **Introduction**

It has been known for some time [1,2,3,4] that the moduli spaces of some Calabi–Yau manifolds meet along boundary components that correspond to certain singular manifolds. Differently put: certain singular Calabi–Yau manifolds can be approached as limits of deformation classes of topologically distinct manifolds. In [1] M. Reid made a bold conjecture that the parameter space of threefolds with vanishing first Chern class is connected. In [3,5] it was shown that the parameter space of all CICY's is connected. CICY's are complete intersection Calabi-Yau manifolds [2,5,6,7]: Calabi–Yau manifolds that can be realized as complete intersections of polynomials defined on products of projective spaces. The class of CICY's comprises several thousand [8] topologically distinct manifolds corresponding to some 250 pairs of values for the Hodge numbers (h_{11}, h_{21}) and with Euler numbers in the range $-200 \leq \chi \leq 0$. At the time of [3] this was the largest class of Calabi–Yau manifolds that could be systematically constructed. Since then another, perhaps larger, class has been studied. These are manifolds that can be realized by a transverse polynomial (a polynomial p such that p and dp do not simultaneously vanish) in [1] $\mathbb{P}_4^{\mathbf{k}}$, a weighted $\mathbb{P}_4$, with weights $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5)$ [9,10,11]. The authors of [10,11] constructed a list $\mathcal{L}$ of 7555 weight vectors $\mathbf{k}$ corresponding to these manifolds. These weight vectors do not all lead to distinct CY manifolds though there are roughly 2,500 distinct pairs of Hodge numbers. The Euler numbers of this class of manifolds lie in the range $-960 \leq \chi \leq 960$. To our knowledge no Calabi–Yau manifolds are known with Euler numbers outside this range. In virtue of this, we considered it of interest to ask whether the moduli spaces of all the manifolds of the list $\mathcal{L}$ are connected together. The result that we report in this paper is that they are.

Reid conjectured that the moduli spaces of all Calabi–Yau manifolds are connected via conifolds [3], though it would be in accord with the conjecture for the connection to

[1] Strictly speaking, the Calabi–Yau manifolds are embedded in blow ups of $\mathbb{P}_4^{\mathbf{k}}$

be through non-Kähler manifolds with vanishing first Chern class. We have found that, at least within our limited class, it is not neccessary to leave the family of Kähler Calabi–Yau manifolds in order to show that the moduli spaces form a connected web. Though the singular manifolds that connect the moduli spaces are in many cases more singular than conifolds. Since the list contains the quintic threefold, $\mathbb{P}_4[5]$, which is also a CICY, it follows that these moduli spaces are connected also to the web of CICY's.

For the case of CICY's it was possible to show that the moduli spaces of the manifolds were connected by means of an analytic argument. Each CICY is specified by a degree matrix and the authors of [7] have shown that a CICY exists for each such matrix. A conifold transition corresponds to a certain operation on the degree matrix and it is straightforward to see that one can transform any two given degree matrices into one another by a sequence of such moves. The analogous argument for weighted CICY's fails however, at least in its naive form, since there are degree matrices for which the corresponding weighted CICY has terminal singularities (which can not be resolved to give a manifold with $c_1 = 0$). Although there is no general proof that the moduli spaces are connected for weighted CICY's it is clear [12] that many weighted CICY's are connected to the web of CICY's and weighted $\mathbb{P}_4$'s.

Our investigation relies heavily on the toric construction of Calabi–Yau manifolds due to Batyrev [13,14]. For a manifold $\mathcal{M}$ of a degree d, (defined as the vanishing locus of a polynomial p of a degree d in a weighted $\mathbb{P}_4$) denote by $(x_1, \ldots, x_5)$ the homogeneous coordinates of the projective space and by $\mathbf{x}^{\mathbf{m}}$ the monomial $x_1^{m_1} \ldots x_5^{m_5}$. The set of all possible exponents $\mathbf{m}$ corresponding to a degree $d = k_1 + \ldots + k_5$ polynomial forms the Newton polyhedron of $\mathcal{M}$. Batyrev observed that in many cases the Newton polyhedron of a manifold of the list $\mathcal{L}$ has a certain property termed reflexivity. In [15] it was checked that in fact all the manifolds of the list have this property. Now a converse obtains: if a polyhedron Δ is reflexive then a Calabi–Yau manifold $\mathcal{M}_\Delta$ may be constructed from Δ. It may happen that a given reflexive polyhedron contains a subsets of points that themselves form a reflexive polyhedron, δ. When this happens the moduli spaces of $\mathcal{M}_\Delta$ and $\mathcal{M}_\delta$ intersect [16]. This is so because the polynomial of $\mathcal{M}_\delta$ contains a subset of the monomials

of that for $\mathcal{M}_\Delta$ and so can be deformed continuously by letting some of the coefficients in the polynomial for $\mathcal{M}_\Delta$ tend to zero. It follows also from a recent theorem of Hayakawa [17] that the distance from the generic smooth manifold $\mathcal{M}_\Delta$ or $\mathcal{M}_\delta$, to the singular manifolds $\mathcal{M}^\sharp$ (that correspond to the intersection of the two moduli spaces) is finite.

Recently, a possible explanation of a manner in which string theory unifies the moduli spaces of many (possibly all) Calabi–Yau vacua was discovered [18,19]. It is argued that black hole condensation can occur at conifold singularities in the moduli space of type IIB Calabi–Yau string vacua, and in some cases this condensate signals a smooth transition to a new, topologically different Calabi–Yau vacuum. The extent to which it is possible to give a similar physical interpretation to the more complicated singularities that arise here is an interesting open question.

The paper is organized as follows. In §2 we try to motivate intuitively how we can deduce the connectivity of moduli spaces from the nesting of polyhedra. In §3 we provide some of the mathematical details to support this idea. After recalling some facts about how reflexive polyhedra are constructed, we establish that if one polyehedron contains another, then the moduli spaces of the corresponding families of Calabi–Yau manifolds are connected. We give also a concrete example of this procedure. In §4 we provide some of the details about the algorithm used to show the connectedness of the moduli spaces associated with $\mathcal{L}$. Finally, in the appendix we provide a list of polyhedra that realize the connectedness as described in this article.

We are grateful to D. Morrison for initially suggesting this problem to us.

Just prior to the submission of this article, we were made aware of similar results reported in [20].

2. Nesting of Reflexive Polyhedra

Newton polyhedra are important since they will allow us to describe how the moduli spaces meet. Suppose that a Newton polyhedron Δ_1 from $\mathcal{L}$ contains a reflexive polyhedron Δ_2 as a subpolyhedron. Then the rough idea is that Δ_2 is obtained from Δ_1 by setting to zero certain coefficients in the polynomial p_1 of Δ_1. This is clearly a continuous operation. In this way we see that the moduli space of the Calabi–Yau manifolds corresponding to Δ_1 intersects the moduli space of Calabi–Yau manifolds corresponding to Δ_2. If there is a third polyhedron Δ_3 that also contains Δ_2

$$\Delta_1 \supset \Delta_2 \subset \Delta_3 \tag{2.1}$$

then the moduli space of the manifolds corresponding to Δ_3 also intersects the moduli space of those corresponding to Δ_2 and the three moduli spaces are connected.

The moduli spaces corresponding to all Δ's that contain a common subpolyhedron are connected. If we denote by F_δ the "family" of moduli spaces corresponding to polyhedra that contain a given subpolyheron δ, and if any polyhedron of a family F_{δ_1} has a subpolyhedron in common with any polyhedron of a family F_{δ_2}, then the two families F_{δ_1} and F_{δ_2} are connected.

There are however some aspects of this process that need to be explained. The apparent difficulty is that Δ_1 corresponds to a family of hypersurfaces in a weighted projective space, with weight vector $\mathbf{k}_1$ say, and Δ_2 to a hypersurface in a weighted projective space with a different weight $\mathbf{k}_2$, say. This difficulty is only apparent; the essential point is that, given a reflexive polyhedron Δ, a deformation class of Calabi–Yau manifolds may be constructed from Δ, such that the generic manifold in the class is smooth. It is important that the reflexivity of Δ is sufficient for this to be true. Δ does not have to correspond to a member of $\mathcal{L}$. This is fortunate, since some of the polyhedra that we use to prove the connectivity of the moduli space, do not belong to $\mathcal{L}$. The resolution of the apparent

difficulty concerning the different weight vectors, is in essence just a point made above about the smoothness of the generic Calabi–Yau manifold corresponding to a reflexive polyhedron. By setting to zero the coefficients that take us from Δ_1 to Δ_2, we obtain a singular manifold $\mathcal{M}_1^\sharp$, in $\mathbb{P}_4^{\mathbf{k}_1}$. This same singular space can be realized as the limit of a hypersurface in $\mathbb{P}_4^{\mathbf{k}_2}$. It is important to note that a family of smooth Calabi–Yau manifolds are in one–to–one correspondence with a dual[2] pair of reflexive polyhedra, (Δ_1, ∇_1), say. There is, similarly, a family for (Δ_2, ∇_2). Now, $\Delta_1 \supset \Delta_2$ and it is important that duality reverses the inclusion for the dual polyhedra so that $\nabla_1 \subset \nabla_2$. The two families meet in manifolds $\mathcal{M}^\sharp$, which may be considered to correspond to the non–dual pair (Δ_2, ∇_1).

We express this as a diagram:

$$\begin{array}{ll}
\mathcal{M}_1 : & (\ \Delta_1, \nabla_1\ ) \\
& \quad\ \downarrow \\
\mathcal{M}^\sharp : & (\ \Delta_2, \nabla_1\ ) \\
& \qquad\quad \uparrow \\
\mathcal{M}_2 : & (\ \Delta_2, \nabla_2\ )
\end{array} \tag{2.2}$$

where the arrows denote specializations of either the polynomial corresponding to $\mathcal{M}_1$ or the mirror polynomial corresponding to $\mathcal{M}_2$.

This diagram allows a nice interpretation. We may think of the singularization $(\Delta_1, \nabla_1) \longrightarrow (\Delta_2, \nabla_1)$ as being due to the specialization of the polynomial which forces the hypersurface to be singular. Alternatively, we can singularize via $(\Delta_2, \nabla_2) \longrightarrow (\Delta_2, \nabla_1)$. We will see below that the process $\nabla_2 \longrightarrow \nabla_1$ can be thought of as a singularization of the embedding space. Speaking loosely, we can say that we achieve the same effect by singularizing the hypersurface while leaving alone the embedding space of one manifold, or by leaving the hypersurface alone and singularizing the embedding space of the other manifold.

The next section, which is rather technical, shows that the operations that correspond to reversing the arrows in (2.2) render the manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ smooth.

[2] The dual of a convex polyhedron is defined in §3.1

3. Calabi-Yau Manifolds in Projective Varieties

3.1. Generalities

We consider Calabi–Yau manifolds as hypersurfaces in toric varieties that are toric deformations of weighted projective spaces. These varieties, as well as the relations between them, will be described in terms of toric geometry. To this end we briefly review the toric construction of projective varieties [4,13,14,15].

We start with a weighted projective space $\mathbb{P}_4^{\mathbf{k}}$, and consider the family of homogeneous polynomials $p = p(x_1, \ldots, x_5)$ of degree $d = \sum_{i=1}^{5} k_i$. As in §1, we associate a vector of exponents to each monomial and write $\mathbf{x}^{\mathbf{m}}$ for $x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} x_5^{m_5}$. Thus, a general homogeneous polynomial of degree d has the form:

$$p = \sum_{\substack{\mathbf{k}\cdot\mathbf{m}=d,\\ m_i \geq 0}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \ .$$

Each degree vector $\mathbf{m}$ can be regarded as a point in $\mathbb{Z}^5 \otimes \mathbb{R}$, and the convex hull of these points forms the Newton polyhedron $\Delta(\mathbf{k})$ of p, though to avoid encumbering the notation, we will largely suppress the dependence on $\mathbf{k}$ in the following. Because of the relation between d and $\mathbf{k}$ the only integral point inside Δ is $\mathbf{1} \stackrel{\text{def}}{=} (1,1,1,1,1)$.

The Newton polyhedron lives in the four dimensional sublattice of $\mathbb{Z}^5$ defined by:

$$\Lambda = \{\mathbf{m} \in \mathbb{Z}^5 \mid \mathbf{k}\cdot\mathbf{m} = d\}$$

or, after translating the origin to the interior point $\mathbf{1}$ and setting $\mathbf{m}' = \mathbf{m} - \mathbf{1}$, by:

$$\Lambda = \{\mathbf{m}' \in \mathbb{Z}^5 \mid \mathbf{k}\cdot\mathbf{m}' = 0\} \ .$$

We denote by V the dual lattice to Λ. The corresponding vector spaces in which these lattices are embedded are $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ and $\mathrm{V}_{\mathbb{R}} = \mathrm{V} \otimes \mathbb{R}$. Inside these vector spaces the Newton polyhedron and its dual are defined as:

$$\Delta(\mathbf{k}) = \text{ convex hull of } \quad \{\mathbf{m}' \in \Lambda(\mathbf{k}) \mid m_i' \geq -1, i = 1, \ldots, 5\}$$

$$\nabla(\mathbf{k}) = \{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{y}\rangle \geq -1, \forall \mathbf{y} \in \Delta\}$$

All the Newton polyhedra constructed this way from $\mathcal{L}$, have been checked to be reflexive [15]. Reflexivity means (in geometrical terms) that:

1. Δ has integer vertices
2. There is one and only one interior point in Δ
3. The equation of any face of codimension 1, which we write as $c_1 y_1 + \cdots + c_4 y_4 = 1$, has coefficients $c_1, \ldots, c_4$ that are integers with no common factor.

The reflexivity condition is important since a polyhedron is reflexive if and only if it is the support of global sections of the anticanonical sheaf on a Gorenstein Fano variety $\mathcal{V}_{(\Sigma,\nabla)}$ [14]. Any such section is a linear combination of monomials that correspond to integral points in Δ. This variety is a blow up of the $\mathbb{P}_4^{\mathbf{k}}$ that we started with. A hypersurface in such a variety, the zero locus of homogeneous polynomials of fixed degree, admits a Calabi–Yau resolution. The fan of the embedding variety Σ is the fan over a triangulation of the faces of the dual polyhedron ∇.

There are two ways of constructing refinements of the fan Σ. One is to take all rays supported by integral points $x \in \nabla \cap \mathrm{V}$ in the dual polyhedron and, given a triangulation, to construct the respective fan (we may still be left with cones of volume [3] greater than 1, but we are guaranteed that the Calabi–Yau hypersurfaces are going to be smooth). In this case we have a crepant [4] morphism of toric varieties, $\phi : \mathcal{V}_{(\Sigma',\nabla)} \longrightarrow \mathcal{V}_{(\Sigma,\nabla)}$. When we refine a fan we say that we blow up the associated variety. The inverse procedure will be called a blow down. The variety $\mathcal{V}_{(\Sigma',\nabla)}$ is obtained from $\mathcal{V}_{(\Sigma,\nabla)}$ by toric resolutions that do not affect the canonical class: the family of functions that may be used to define a Calabi–Yau hypersurface is not changed in the process. The other way of refining the fan is to choose additional rays that do not correspond to integral points in ∇. These rays intersect the facets of ∇ in non-integral points. In some cases the primitive integer

[3] If all maximal cones of the fan have volume equal to 1, then the variety is smooth.

[4] A map is crepant if it preserves the canonical class: in both varieties, Calabi–Yau hypersurfaces can be defined in terms of the same set of functions.

vectors along these rays may define a new polyhedron $\hat{\nabla}$ which is also reflexive. Even though the canonical class of the variety has been affected in the process (there is a new set of functions that we can use in defining Calabi–Yau hypersurfaces) the reflexivity of the polyhedron $\hat{\nabla}$ associated with the new variety guarantees the existence of Calabi–Yau hypersurfaces. We claim that whenever a variety $\mathcal{V}_{(\Sigma,\nabla)}$ admits such a noncrepant blow up there is an isomorphism between the induced deformation of a subset of the Calabi–Yau family originally embedded in $\mathcal{V}_{(\Sigma,\nabla)}$ and the Calabi-Yau family of $\mathcal{V}_{(\hat{\Sigma},\hat{\nabla})}$.

3.2. Connectivity of Moduli Spaces

As mentioned in the introduction we aim to show the connectedness of the moduli spaces for the 7555 families of Calabi–Yau manifolds that can be obtained as transverse hypersurfaces in projective varieties [10,11]. The constructive method outlined above tells us that each polyhedron Δ lives in the hyperplane defined by the weight vector $\mathbf{k}$. The sublattice Λ generated by this hyperplane in $\mathbb{Z}^5$ has relative volume $\sqrt{\sum_{i=1}^{5} k_i^2}$. Once we find an appropriate basis for Λ we express the coordinates of all integral points with respect to it. Note that there are an infinite number of ways to choose a basis for any lattice of dimension greater then 1 and we will identify polyhedra $\Delta(\mathbf{k})$ and $\Delta(\mathbf{k}')$ if there is a $GL(4,\mathbb{Z})$ bijection between them. Because the group $GL(4,\mathbb{Z})$ is volume preserving we do not change the structure of the fans supported by Δ. Each maximal cone will preserve its volume as well as the number of integral points it contains. This shows that the two varieties are indeed isomorphic, $\mathcal{V}_{(\Sigma,\nabla(\mathbf{k}))} \cong \mathcal{V}_{(\Sigma',\nabla(\mathbf{k}'))}$, and this is also true for the respective families of Calabi–Yau hypersurfaces. When we say that $\Delta_2 \subset \Delta_1$, we mean that there is a transformation that matches all the points in Δ_2 to a subset of points in Δ_1. For example, if the point a in Δ_1 corresponds to the monomial $\mathbf{x}^{\mathbf{m}}$ and the point b in Δ_2 corresponds to the monomial $\mathbf{y}^{\mathbf{n}}$, then identifying a and b implies $\mathbf{x}^{\mathbf{m}} = \mathbf{y}^{\mathbf{n}}$.

Let us now examine more closely the consequences of the inclusion $\Delta_2 \subset \Delta_1$.

We have:

$$\begin{array}{ccc}
\mathcal{V}_{(\Sigma_2,\nabla_2)} & & \mathcal{V}_{(\Sigma_1,\nabla_1)} \\
\updownarrow & & \updownarrow \\
\Delta_2 & \subset & \Delta_1 \\
\updownarrow & & \updownarrow \\
\nabla_2 & \supset & \nabla_1
\end{array}$$

where ∇_2 is the support of a noncrepant toric deformation of $\mathcal{V}_{(\Sigma_1,\nabla_1)}$ (Σ_2 is a refinement of Σ_1), such that

$$\mathcal{V}_{(\Sigma_2,\nabla_2)} \xrightarrow{\phi} \mathcal{V}_{(\Sigma_1,\nabla_1)}$$

is a proper birational morphism of toric varieties.

Consider $\mathcal{M}_2 \subset \mathcal{V}_{(\Sigma_2,\nabla_2)}$ to be a generic hypersurface. The points in $\nabla_2 \cap \mathrm{V}_2$ correspond to a subset of monomials in $\nabla_1 \cap \mathrm{V}_1$ that define a certain hypersurface $\mathcal{M}_1^\sharp \subset \mathcal{V}_{(\Sigma_1,\nabla_1)}$. The crucial observation is that the pullback of $\mathcal{M}_1^\sharp$ to $\mathcal{V}_{(\Sigma_2,\nabla_2)}$ under ϕ is isomorphic to $\mathcal{M}_2$ [15]. Otherwise said, if $\mathcal{V}_{(\Sigma_1,\nabla_1)}$ is blown up to $\mathcal{V}_{(\Sigma_2,\nabla_2)}$, then the induced deformation of $\mathcal{M}_1^\sharp$ is isomorphic to $\mathcal{M}_2$.

$$\begin{array}{ccc}
\mathcal{M}_2 & \leftarrow \text{birational map} \rightarrow & \mathcal{M}_1^\sharp \\
\cap & & \cap \\
\mathcal{V}_{(\Sigma_2,\nabla_2)} & \rightarrow \text{non crepant blowup} \rightarrow & \mathcal{V}_{(\Sigma_1,\nabla_1)}
\end{array}$$

What do the inclusions (2.1), $\Delta_1 \supset \Delta_2 \subset \Delta_3$, tell us? There is a special sub-family of hypersurfaces $\mathcal{M}_1^\sharp \subset \mathcal{V}_{(\Sigma_1,\nabla_1)}$ that can be "deformed" into the full family of hypersurfaces $\mathcal{M}_2 \subset \mathcal{V}_{(\Sigma_2,\nabla_2)}$, that in turn can be "deformed" into a special sub-family $\mathcal{M}_3^\sharp \subset \mathcal{V}_{(\Sigma_3,\nabla_3)}$. We remark that the resolution of either $\mathcal{M}_1^\sharp$ or $\mathcal{M}_3^\sharp$ allows us to control all the polynomial deformations of $\mathcal{M}_2$.

3.3. Illustration of the Method

We want to show how the above analysis applies to a pair of varieties. Consider the manifold $\mathbb{P}^{(24,51,133,416,624)}[1248]^{h_{11}=214}_{h_{21}=10}$. There are 18 points in the polyhedron and 12 of these are shared with the polyhedron associated with $\mathbb{P}^{(54,56,151,522,783)}[1566]^{h_{11}=251}_{h_{21}=5}$. We

are going to look at the vertices of the interior polyhedron and see how they relate to points of the exterior polyhedron. A basis for the sublattice defined by $\mathbf{k}_1 = (24, 51, 133, 416, 624)$ is:

$$\begin{aligned}
\mathbf{v}_1 &= (-26,\ \ 0,\ \ 0,\ \ 0,\ \ 1)\\
\mathbf{v}_2 &= (-25,\ \ 1,\ \ 1,\ \ 1,\ \ 0)\\
\mathbf{v}_3 &= (-52,\ \ 0,\ \ 0,\ \ 3,\ \ 0)\\
\mathbf{v}_4 &= (-17,\ \ 8,\ \ 0,\ \ 0,\ \ 0)
\end{aligned}$$

The points that correspond to the vertices of the inside polyhedron are

$$\begin{aligned}
\mathbf{a}_1 &= (\ \ 1, -1,\ \ 0,\ \ 0) = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 2)\\
\mathbf{a}_2 &= (-1, -1,\ \ 1,\ \ 0) = (\ \ 0,\ \ 0,\ \ 0,\ \ 3,\ \ 0)\\
\mathbf{a}_3 &= (-1,\ \ 8, -3, -1) = (\ \ 0,\ \ 1,\ \ 9,\ \ 0,\ \ 0)\\
\mathbf{a}_4 &= (-1,\ \ 2, -1,\ \ 0) = (\ 29,\ \ 3,\ \ 3,\ \ 0,\ \ 0)\\
\mathbf{a}_5 &= (-1, -1,\ \ 0,\ \ 3) = (\ \ 1,\ 24,\ \ 0,\ \ 0,\ \ 0)
\end{aligned}$$

The vectors on the extreme right with 5 components are the $\mathbf{m}$ vectors from which we can read the monomials to which they correspond. Turning now to the second manifold, we find that a basis for the sublattice defined by $\mathbf{k}_2 = (54, 56, 151, 522, 783)$ is:

$$\begin{aligned}
\mathbf{w}_1 &= (\ 29,\ \ 0,\ \ 0,\ \ 0, -2)\\
\mathbf{w}_2 &= (\ \ 1,\ \ 1,\ \ 1,\ \ 1, -1)\\
\mathbf{w}_3 &= (\ \ 0,\ \ 0,\ \ 0,\ \ 3, -2)\\
\mathbf{w}_4 &= (\ 10,\ \ 9,\ \ 0,\ \ 1, -2)
\end{aligned}$$

The vertices of the interior polyhedron are:

$$\begin{aligned}
\mathbf{b}_1 &= (\ \ 0, -1,\ \ 0,\ \ 0) = (\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 2)\\
\mathbf{b}_2 &= (\ \ 0, -1,\ \ 1,\ \ 0) = (\ \ 0,\ \ 0,\ \ 0,\ \ 3,\ \ 0)\\
\mathbf{b}_3 &= (\ \ 0,\ \ 9, -3, -1) = (\ \ 0,\ \ 1,\ 10,\ \ 0,\ \ 0)\\
\mathbf{b}_4 &= (-1, -1, -1,\ \ 3) = (\ \ 1,\ 27,\ \ 0,\ \ 0,\ \ 0)\\
\mathbf{b}_5 &= (\ \ 1, -1,\ \ 0,\ \ 0) = (\ 29,\ \ 0,\ \ 0,\ \ 0,\ \ 0)
\end{aligned}$$

Define $\mathbf{A}_i^j = (\mathbf{a}_i)^j$ and $\mathbf{B}_i^j = (\mathbf{b}_i)^j$. Then $\mathbf{A} = \mathbf{B}\mathbf{T}$ where

$$\mathbf{T} = \begin{pmatrix} -2 & 0 & 0 & 3 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}$$

If we take the space $\mathbb{P}^{(24,51,133,416,624)}[1248]^{h_{11}=214}_{h_{21}=10}$ to have homogeneous coordinates x_i and the space $\mathbb{P}^{(54,56,151,522,783)}[1566]^{h_{11}=251}_{h_{21}=5}$ to have homogeneous coordinates y_i then the correspondence $\mathbf{a}_i \leftrightarrow \mathbf{b}_i$ gives the following identification of monomials:

$$\begin{aligned} x_5^2 &= y_5^2 \\ x_4^2 &= y_4^2 \\ x_2 x_3^9 &= y_2\, y_3^{10} \\ x_1^{29}\, x_2^3\, x_3^3 &= y_1\, y_2^{27} \\ x_1\, x_2^{24} &= y_1^{29} \end{aligned}$$

which gives in turn the birational map between the two varieties:

$$\begin{aligned} y_1 &= x_1^{1/29}\, x_2^{24/29} \\ y_2 &= x_1^{840/783}\, x_2^{63/783} x_3^{1/9} \\ y_3 &= x_1^{-84/783}\, x_2^{72/783} x_3^{8/9} \\ y_4 &= x_4 \\ y_5 &= x_5 \end{aligned}$$

The map is one to one despite appearances. This must be the case in virtue of our analysis in §3.2. and may be explicitly checked by using the scaling properties of the two sets of coordinates. To conclude, we have a sub-family of hypersurfaces in the variety described by the exterior polyhedron given by the zero locus of the polynomial

$$\begin{aligned} p^\sharp = {} & x_1x_2x_3x_4x_5 + x_5^5 + x_4^3 + x_2x_3^9 + x_1^2x_2^2x_3^2x_4^2 + x_1^{29}x_2^3x_3^3 \\ & + x_1^3x_2^3x_3^3x_5 + x_1^4x_2^4x_3^4x_4 + x_1^6x_2^6x_3^6 + x_1^{10}x_2^9x_3x_4 + x_1^{12}x_2^{11}x_3^3 + x_1x_2^{24} \end{aligned}$$

that can be blown up to a generic hypersurface in the variety associated with the interior polyhedron given by the general polynomial

$$\begin{aligned} p^\flat = {} & y_1y_2y_3y_4y_5 + y_5^2 + y_4^3 + y_2y_3^{10} + y_1^2y_2^2y_3^2y_4^2 + y_1y_2^{27} \\ & + y_1^3y_2^3y_3^3y_5 + y_1^4y_2^4y_3^4y_4 + y_1^6y_2^6y_3^6 + y_1^{10}y_2^9y_4 + y_1^{12}y_2^{11}y_3^2 + y_1^{29} \end{aligned} .$$

4. **The Computation**

In principle, the task of determining all of the reflexive sub-polyhedra (RSP's) of a given reflexive polyhedron (RP) is straightforward. Suppose the RP in question has N points. One can then first look for all RSP's with $N-1$ points. There are $N-1$ candidates (recall that the interior point of the RP must also be the interior point of the RSP). For each candidate, one would have to determine if it is reflexive. This requires identifying the faces, and determining their equations. In general, there are $\binom{N-1}{M-1}$ candidates for RSPs with M points. So, the total number of candidates one would have to consider is:

$$\sum_{M=6}^{N-1} \binom{N-1}{M-1} \approx 2^{N-1}.$$

Since many RP's have $N > 200$, this is not feasible. While this method is complete, and would have to be followed in order to establish that a group of RP's are *not* directly connected, it is thankfully not necessary if one is trying to establish their connectedness. One might imagine that it might happen that all of the RP's under consideration contain a particular RSP, and connectedness would follow immediately, irrespective of any other RSP's that they might contain. This turns out not to be the case, but is the spirit in which we have attacked the problem.

There is no such RSP. To see this is simple. There do exist reflexive polyhedra with 6 points. These are simplices in four dimensions (recall that there is always one interior point), and they cannot contain any RSP's, hence they themselves would have to be the magical RSP's. However, there are three inequivalent RP's with 6 points. Although the simplest guess fails, nevertheless one might hope to restrict one's search for RSP's to some simple objects, such as simplices.

Our initial strategy was to identify the reflexive simplices within each of the 7555 RP's. In fact we should identify those reflexive simplices which themselves contain no reflexive

simplices. We will refer to such objects as 5–vertex irreducible simplices, that is they are simplices that contain no 5–vertex RSP's. In general, any reflexive polyhedron which does not contain any n-vertex reflexive polyhedra (apart from itself) is an n-vertex irreducible polyhedron. The combinatoric barrier is now reduced from 2^{N-1} to $\binom{N-1}{5}$, which is a good deal more manageable. In fact, it was a relatively quick matter to decompose all RP's with $N \leq 120$ into 5-vertex irreducible simplices. Very early in this procedure (i.e. after decomposing a small fraction of these RP's), a set of 41 5-vertex irreducible simplices was generated, ranging in size from $N = 6$ to $N = 26$ points. It turned out that these were the only ones that were generated by this procedure, though we have no proof that others do not exist. Among them are 18 that are not on the list of 7555 that we started from. Being simplices, these must correspond to Fermat polynomials, but since they are not all in the list, they must in fact correspond to manifolds of the form $\{p = 0\}/G$, where p is Fermat and G is a group of automorphisms.

Establishing the connectedness of all of those RP's that contain one or more of these simplices is then a matter of showing that these 41 simplices are connected to each other. Let us denote the set of 5-vertex irreducible simplices contained in the ith RP as $\mathbf{V}_i^5$. We define $\mathbf{C}_1^5 = \mathbf{V}_1^5$. In general:

$$\mathbf{C}_i^5 = \mathbf{C}_{i-1}^5 \cup \delta_{i-1,i} \ , \quad \text{where} \quad \delta_{i,j} = \begin{cases} \emptyset & \text{if } \mathbf{C}_i^5 \cap \mathbf{V}_j^5 = \emptyset \\ \mathbf{V}_j^5 & \text{if } \mathbf{C}_i^5 \cap \mathbf{V}_j^5 \neq \emptyset \end{cases}$$

If for any i, $\mathbf{C}_i^5$ contains all 41 of the 5-vertex irreducible simplices, then their connectedness has been established. This procedure, which is a sufficient but not necessary condition, can be described as follows. For each polyhedra with $N \leq 120$, we have a list of which of the 41 reflexive simplices that it contains, $\mathbf{V}_i^5$. Clearly, all of the simplices that are contained in $\mathbf{V}_1^5$ are connected, since they are all connected to the first polyhedron. So this is our initial list of connected simplices. Now, successively examine each of the $\mathbf{V}_i^5$. Whenever there is a simplex in $\mathbf{V}_i^5$ that is also in our list of connected simplices, we add *all* of the simplices in $\mathbf{V}_i^5$ to our list. As soon as this list contains all 41 simplices, we have shown that they are connected. This is indeed the case, hence all those RP's with $N \leq 120$ and $\mathbf{V}_i^5 \neq \emptyset$, which amounts to 6133 RP's, are connected.

For those RP's with $N > 120$, we take advantage of the apparent completeness of the list of 5-vertex irreducible simplices. So instead of finding *every* 5-vertex irreducible simplex in these larger objects, we stop our search as soon as we find *one*. This approach succeeds, in that all of these larger RP's contain at least one of the 41 previously found 5-vertex irreducible simplices. This accounts for a further 1280 of the RP's.

We were left with 142 RP's that do not contain any reflexive simplices. As they are relatively small ($N \leq 28$), it was feasible to try and search for RSPs that had 6 vertices, and were themselves 6-vertex irreducible. Applying this procedure, we found that 132 of the remainder contained 6-vertex irreducible 6-vertex RSPs, and that 10 did not.

In order to see if these 132 RP's were connected to the 6133 RP's that were already known to be interconnected, it was necessary to make sure that all of the 6-vertex irreducible 6-vertex RSPs that were generated by the former list, were also generated by some subset (hopefully a small one) of the latter list. In practice, it was easy to establish this for 123 of them. The remaining 9 are difficult cases.

We then attempted to decompose these 9, plus the 10 RP's that contained neither reflexive simplices, nor reflexive 6-vertex polyhedra, into 7-vertex irreducible 7-vertex RSPs. Seventeen of them yielded to this procedure, and were easily connected to the other 7536 RP's. This leaves us with two polyhedra. They correspond to $\mathbb{P}_4^{(21,24,82,111,119)}[357]$ and $\mathbb{P}_4^{(18,21,58,77,78)}[252]$. They each contain the same 6-vertex irreducible 6-vertex RSP – one which we could not easily locate in another RP. Let us call it Π.

In order to take care of these last two polyhedra, we make use of the elementary observation made previously, that if Δ_2 and Δ_1 are reflexive polyhedra, and $\Delta_2 \subset \Delta_1$, then $\nabla_1 \subset \nabla_2$, where ∇_2 and ∇_1 are the dual polyhedra of Δ_2 and Δ_1. First, we established that the dual of Π was one of the RP's from our original list. It corresponds to $\mathbb{P}_4^{(1,1,4,6,6)}[18]$. Thus the duals of whatever RSPs are contained within this RP, contain Π. In fact, the RP corresponding to $\mathbb{P}_4^{(1,1,4,6,6)}[18]$ contains a simplex whose dual polyhedron corresponds to $\mathbb{P}_4^{(1,1,1,6,9)}[18]$. So, we learn that $\mathbb{P}_4^{(21,24,82,111,119)}[357]$ and $\mathbb{P}_4^{(18,21,58,77,78)}[252]$ each contain an RSP that is also contained in the RP corresponding

to $\mathbb{P}_4^{(1,1,1,6,9)}[18]$. Furthermore, the RP corresponding to $\mathbb{P}_4^{(1,1,1,6,9)}[18]$ is one of the RP's that had previously been connected via 5-vertex irreducible simplices.

We have thus succeeded in establishing that the moduli spaces of all three dimensional CICYs and all transverse polynomials in four dimensional weighted projective spaces are interconnected.

Acknowledgements

We gratefully acknowledge fruitful conversations with S. Katz, S. Keel, D. Morrison and D. Saltman. This work was supported by NSF Grants PHY95-11632 and PHY94-07194 and the Robert A. Welch Foundation. P. Candelas and M. Mandelberg would like to thank SBITP for hospitality while part of this work was being done.

A. Appendix: The n-Vertex Irreducible RP's that Connect the Manifolds

Here we give some information regarding those polyhedra which we have used to connect the moduli spaces of all the manifolds of the list $\mathcal{L}$. For each polyhedron, the following information is given: the number of points it contains, its associated hodge numbers, and, in the case that it corresponds to one of the polyhedra corresponding to one or more of the spaces in $\mathcal{L}$, the weights of those spaces, enclosed in square brackets. Instead, if it does not correspond to any entry on $\mathcal{L}$, the vertices of the polyhedron are given, enclosed in parentheses. There are three tables, one for each n-vertex irreducible n-vertex RP's with $n = 5, 6$ and 7.

Table 1: 5-vertex irreducible 5-vertex reflexive polyhedra

Pts	h_{11}	h_{21}	Vertices/Weights
6	103	1	[52,60,63,75,125] [48,50,60,63,79]
6	101	1	[41,48,51,52,64] [51,60,64,65,80]
6	21	1	(0,1,-1,0) (-2,1,1,-1) (-1,-1,0,2) (0,0,-1,0) (3,-1,1,-1)
7	149	1	[75,84,86,98,343] [43,48,56,98,147] [48,49,56,86,153] [42,43,49,75,134]
7	145	1	[73,80,90,162,405] [64,72,73,115,324] [40,45,73,81,166] [40,45,63,64,148]
7	112	4	[42,48,57,109,128]
7	86	2	[27,36,49,56,84] [27,36,41,52,60]
7	89	5	[36,40,89,99,132]
7	38	2	(1,-1,1,-1) (-2,-1,1,1) (0,-1,-1,2) (-1,-1,0,0) (0,2,-1,0)
8	165	3	[60,66,74,163,363] [40,44,74,121,205] [36,40,66,89,165]
8	128	2	[25,30,66,88,121] [25,30,54,82,109] [30,41,50,54,125]
8	103	7	[30,39,84,127,140]
8	101	5	(-1,0,0,1) (-1,0,1,0) (-1,1,0,0) (-1,0,0,0) (11,-1,-4,-2)
8	60	6	(0,-1,0,1) (2,-1,0,0) (0,-1,1,0) (0,-1,0,0) (-3,5,-1,-1)
8	63	3	(0,-1,1,0) (-2,5,-1,-2) (0,-1,0,1) (0,-1,0,0) (2,-1,0,1)
8	31	13	(-3,2,0,1) (0,-1,3,-1) (-1,1,-1,1) (0,-1,0,0) (4,0,-2,-1)
9	148	4	[24,28,83,90,135] [24,28,77,78,129] [24,42,52,71,147]
9	83	3	(0,-1,0,0) (0,-1,1,0) (2,-1,0,0) (-1,-1,0,1) (-1,7,-4,-1)
9	75	3	(2,1,-1,-1) (-1,0,0,0) (-1,0,0,1) (0,-1,-1,2) (1,-2,6,-1)
9	55	7	(0,-1,1,0) (-1,1,-1,1) (-2,3,1,-2) (0,-1,0,0) (2,-1,0,0)
9	43	3	(3,0,0,-1) (1,-1,1,0) (1,0,2,-1) (0,-1,0,0) (-5,5,-3,2)
9	29	5	(-1,0,-1,2) (-1,1,-1,0) (-1,0,-1,0) (3,-1,1,-1) (-1,1,3,-2)
10	272	2	[91,96,102,578,867] [64,68,91,355,578] [48,51,91,289,388] [36,51,64,187,274]
10	143	7	[40,45,143,152,380]
10	105	3	[33,36,40,89,198]

Table 1(cont'd): 5-vertex irreducible 5-vertex reflexive polyhedra

Pts	h_{11}	h_{21}	Vertices/Weights
10	21	9	(2,-1,0,0) (-1,-1,-1,2) (-1,-1,1,0) (-1,-1,1,1) (-1,5,-1,-3)
10	17	21	(2,-1,-1,0) (0,0,-1,0) (-1,3,-1,0) (-1,-1,-1,1) (0,-1,4,-1)
11	164	8	(1,-1,0,0) (0,-1,1,0) (0,-1,0,0) (-1,5,-3,-1) (-2,-1,0,4)
11	131	11	[24,33,138,173,184]
11	69	9	(-1,0,0,1) (-1,1,0,0) (-1,-2,4,0) (-1,0,0,0) (5,-2,-2,-1)
11	59	11	(-1,0,1,0) (-1,2,-1,0) (0,-1,0,1) (0,-1,0,0) (6,-1,0,-1)

Pts	h_{11}	h_{21}	Vertices/Weights
11	243	3	[24,33,92,173,230] [24,44,69,161,254] [24,44,63,155,242]
12	251	5	[54,56,151,522,783] [30,56,87,290,407]
13	82	10	(2,-1,1,-1) (-1,-1,1,1) (-1,1,0,0) (0,-1,0,0) (5,-1,-5,3)
13	35	19	(1,-1,0,0) (-1,-1,1,0) (-1,-1,3,0) (-1,-1,0,1) (-1,7,-4,-1)
14	271	7	[48,51,181,560,840] [36,51,140,403,630]
15	227	11	[30,38,234,283,585] [20,38,156,195,371]
15	103	7	[18,20,57,85,180]
17	321	9	[42,46,241,658,987] [24,46,141,376,541]
21	131	11	[15,16,93,116,240]
26	491	11	[41,42,498,1162,1743] [36,41,421,996,1494] [28,41,332,761,1162] [21,41,249,581,851]

Table 2: 6-vertex irreducible 6-vertex reflexive polyhedra

Pts	h_{11}	h_{21}	Vertices/Weights
7	95	2	[32,42,45,91,105] [26,33,48,75,91] [26,33,45,60,64] [23,28,34,53,55]
7	86	2	[27,29,64,72,96] [22,29,49,50,75] [29,36,75,84,112] [27,28,59,63,75] [21,29,53,56,65] [24,29,50,56,65]
8	132	2	[44,48,63,131,286] [28,39,48,91,158]
8	105	3	(0,-1,0,0) (-1,1,0,0) (5,1,-1,-2) (0,-1,1,0) (2,-1,0,0) (0,-1,0,1)
8	87	3	[22,30,63,95,105]
8	43	7	(-1,-1,0,0) (0,-1,1,0) (1,-1,-1,0) (-1,-1,0,1) (0,1,0,0) (1,4,-1,-1)
8	59	3	(0,-1,1,0) (1,-1,-1,0) (-1,-1,0,1) (-1,-1,1,0) (0,1,0,0) (1,4,-1,-1)
10	152	6	[34,36,122,131,323] [20,34,64,69,153]
10	131	3	[33,40,122,130,325] [20,33,61,65,146]
10	139	5	(-1,1,0,0) (5,0,-1,-1) (-1,0,1,0) (-1,0,0,0) (1,0,-1,1) (-1,-1,-1,3)
10	111	9	(-1,1,0,0) (-1,0,1,0) (-1,0,0,1) (-1,0,0,0) (3,-1,-1,-1) (17,-6,-6,-4)
11	114	6	(-1,0,1,0) (2,-1,1,-1) (-1,1,0,0) (0,-1,0,0) (2,-1,-2,2) (3,-1,-3,2)
11	101	5	[17,24,99,124,132] [17,18,70,93,99]
11	70	4	(0,-1,0,0) (0,-1,0,1) (1,0,0,0) (-1,0,2,0) (0,3,-2,0) (0,6,-3,-1)
13	43	11	(-1,-1,0,0) (3,-1,-2,0) (3,6,-2,-1) (-1,-1,1,0) (1,3,-1,0) (-1,-1,0,1)
14	194	10	(1,-1,0,0) (0,-1,1,0) (0,-1,0,1) (0,-1,0,0) (-5,9,-2,-2) (-16,31,-8,-6)
15	183	7	[23,30,182,220,455] [23,24,141,176,364] [15,23,91,110,216]
15	157	9	(0,-1,0,0) (1,-1,0,0) (-1,-1,1,0) (-1,-1,0,1) (-1,19,-5,-4) (-1,24,-6,-5)
15	166	8	(1,-1,0,0) (-1,-1,1,0) (-1,-1,0,1) (-1,0,0,0) (-1,19,-5,-4) (-1,24,-6,-5)
15	140	10	(-1,2,0,-1) (0,0,1,-1) (0,0,0,-1) (0,-1,0,1) (5,0,0,-1) (5,0,-1,0)

Table 2(cont'd): 6-vertex irreducible 6-vertex reflexive polyhedra

Pts	h_{11}	h_{21}	Vertices/Weights
15	149	9	(-1,2,0,-1) (0,0,1,-1) (0,0,0,-1) (0,-1,0,1) (5,0,0,-1) (5,1,-1,-1)
15	131	11	(-1,2,0,-1) (0,0,0,-1) (0,-1,0,1) (0,-1,1,0) (5,0,0,-1) (5,0,-1,0)
17	295	7	[37,42,216,590,885] [28,37,144,381,590] [21,37,108,295,424]
22	376	10	(-1,0,1,0) (-1,0,0,1) (-1,1,0,0) (-1,0,0,0) (13,0,-4,-2) (59,-1,-20,-8)

Table 3: 7-vertex irreducible 7-vertex reflexive polyhedra

Pts	h_{11}	h_{21}	Vertices/Weights
8	51	3	[20,21,25,26,33]
10	92	5	(0,-1,1,0) (0,-1,0,1) (-1,2,0,-1) (0,-1,0,0) (1,0,0,0) (3,-1,0,0) (4,0,-1,0)
13	152	6	(1,-1,0,0) (-1,-1,1,0) (-1,-1,0,1) (-1,0,0,0) (-1,9,-2,-2) (-1,11,-2,-3) (-1,20,-4,-5)
14	167	7	(-1,1,0,0) (-1,0,0,1) (4,0,-1,-1) (-1,0,0,0) (-1,0,2,0) (0,0,3,-1) (0,-1,5,-1)

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